MATH 4151: History of Mathematics

The Bernoulli Family

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Overview

The Swiss Bernoulli family is probably the family who spawned the most famous mathematicians. They covered almost every field of mathematics of their time—algebra, calculus of variations, differential equations, geometry, infinitesimal calculus, mathematics in physics, number theory to a certain extent, probability theory, and theory of series, to name only a few.

This paper's task is to bring out the most important results and to dignify the achievements of the Bernoullis.

Jakob Bernoulli


(1655[1]-1705)

Jakob Bernoulli (for distinction reasons also Jakob I in order to not confuse him with his grandnephew) was, like his brother Johann, the son of the Swiss spice merchant Niklaus Bernoulli (also Nikolaus).

His main work, *Ars conjectandi* (The Art of Conjecturing), is mainly about probability theory. Therein he defines stochastic

[1]Some sources provide 1654 as his year of birth; in Gregorian calender, indeed he was born on Dec 27, 1654, whereas it is Jan 6, 1655 in Julian calender, which was established in Switzerland in 1584, cf. [Britannica]

> “[...] as the art of evaluating as exactly as possible the probabilities of things, so that in our judgments and actions we can always base ourselves on what has been found to be the best, the most appropriate, the most certain, the best advised; this is the only object of the wisdom of the philosopher and the prudence of the statesman.”

Parts of the *Ars conjectandi* are only a summarization of known results, for some of which he gave his own proofs. However, he provides important theorems he found himself like the weak law of large numbers:

Theorem (Weak Law of Large Numbers)**.** *If $X_1, X_2, \ldots, X_n$ is a sequence of independent and identically distributed random variables, each having the same mean μ and satisfying $\mathbb{E}(X_1^2) < \infty$, then for any $\varepsilon > 0$,*

$$\mathbb{P}(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \varepsilon) \to 0 \text{ as } n \to \infty.$$

The proof is easy if we premise Chebychev’s inequality:

Proof. Since $\text{var}(aX) = \mathbb{E}((aX)^2) - (\mathbb{E}(aX))^2 = \mathbb{E}(a^2X^2) - a^2(\mathbb{E}(X))^2 = a^2\mathbb{E}(X^2) - a^2(\mathbb{E}(X))^2 = a^2\text{var}(X)$ by linearity of $\mathbb{E}$, and $\text{var}(X_i) = \text{var}(X_1)$ for any $i \in \{1, 2, \ldots, n\}$ by assumption (the X_i are identically distributed), and $\text{var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{var}(X_i)$ since the X_i are independent, we have

$$\text{var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} \text{var}(X_i) = \frac{1}{n}\text{var}(X_1).$$

Applying Chebychev’s inequality, we get

$$\mathbb{P}(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \varepsilon) \leq \frac{\text{var}(X_1)}{n\varepsilon^2}.$$

Since ε and $\text{var}(X_1)$ are fixed numbers, the right-hand side tends to 0 as $n \to \infty$. □

The *Ars conjectandi* also contains some strategies how to win various games of luck (cf. [MacTutor]). He also introduces what now are called Bernoulli numbers in there, which occur in many different areas of mathematics, e.g. in the series expansion of $\tan(x)$:

$$\tan(x) = \sum_{k=1}^{\infty} \frac{B_{2k}(-4)^k(1-4^k)}{(2k)!} x^{2k-1} \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2},$$

or in Fermat's Last Theorem, which Ernst Kummer proved in case n is a regular prime, which is the case if (and only if) it does not divide any numerator of the Bernoulli numbers $B_2, B_4, \ldots, B_{n-3}$ (see [Gourdon]). This is a quite strong result since irregular primes are very "rare;" the only primes less than 100 that are not regular are 37, 59, and 67: 37 divides the numerator of B_{32}[2], 59 divides that of B_{44}[3], and 67 finally divides that of B_{58}[4]. There is even a bibliography of Bernoulli numbers; see [Dilcher].

In his *Tractatus de seriebus infinitis*, a treatise on infinite series, he proves that $\sum \frac{1}{n}$ diverges (Theorem XVI of [SerInf]), which he believed considered a new result; it has been proven earlier by Pietro Mengoli (cf. [MacTutor]), though. In Theorem XXIV he presents that $\sum \frac{1}{n^k}$ converges if $k \geq 2$, but only provides 2 as an upper bound, without giving any exact value.

Another famous result due to Jakob Bernoulli is the following:

Theorem (Bernoulli's inequality)**.** *Let* $x \geq -1$. *Then*

$$(1+x)^n \geq 1 + nx \textit{ for any nonnegative integer } n.$$

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[2]which is $7709321041217 = 37 \cdot 208360028141$

[3]namely $27833269579301024235023 = 59 \cdot 471750331852559732797$

[4]because $8448361334888004186204677599403602 1 = 67 \cdot 126094945296835883376189217901546 3$.

The first 499 Bernoulli numbers can be found at

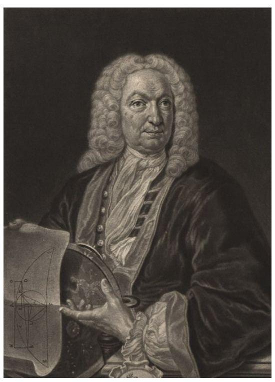
`http://www.gutenberg.org/dirs/etext01/brnll10.txt`

Proof. In case $n = 0$, we have $(1 + x)^0 = 1 \geq 1 + 0 = 1 + 0x$. Assume the inequality holds for $n \geq 0$. Since $1 + x \geq 0$ by assumption, multiplying $(1 + x)^n \geq 1 + nx$ on both sides with $(1 + x)$, which is nonnegative by assumption, gives us

$$(1 + x)^{n+1} = (1 + x)^n(1 + x) = (1 + nx)(1 + x) = 1 + x + nx + nx^2 \geq 1 + (n + 1)x,$$

where the last "$\geq$" is true because n is a nonnegative integer and $x^2 \geq 0$, so $nx^2 \geq 0$. But this is the statement for $(n + 1)$. Hence, the formula is true for $(n + 1)$ if it is true for $n \geq 0$, and, therefore, for every nonnegative integer. □

Johann Bernoulli


(1667-1748)

Although they achieved many successes when they worked together, there was a big rivalry between Jakob and his younger brother: they assigned tasks for each other in mathematical journals like the *Acta Eruditorum*. In there, Johann posed the brachistochrone problem in 1696, which Jakob could solve one year later. When Jakob asked for a generalization, Johann was unable to do so (cf. [MacTutor2]).

One of the major results due to Johann Bernoulli is, what is today misattributed as, l'Hôpital's rule[5]: for differentiable functions f and g the limit of the quotient in case it would be an expression like $0/0$ or ∞/∞ can be acquired by evaluating the limit of the quotient of the derivatives:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

According to [Reich], Johann was the first one to solve the differential equation that nowadays is called Bernoulli equation and was proposed by Jakob Bernoulli in 1695, although other sources state it was Jakob himself who first solved it (cf. [MacTutor]):

$$\dot{y} = p(x)y + q(x)y^{n-1}.$$

By multiplication with y^{-n} on both sides, and change of variables $z = y^{1-n}$ (i.e. $\dot{z} = (1-n)y^{-n}\dot{y}$) results in a first order linear differential equation:

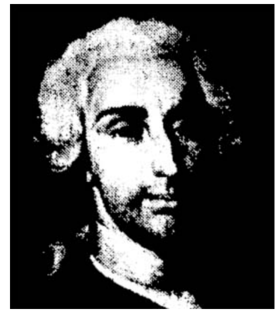
$$\frac{\dot{z}}{1-n} = f(x)z + g(x)$$

[5]according to [MacTutor2], l'Hôpital paid a high rent to Bernoulli after the latter agreed to a contract that guaranteed l'Hôpital to be the only one getting the latest discoverings from Bernoulli and getting problems solved by him

Other family members

Nicolaus (II) Bernoulli (1695-1726)

Daniel Bernoulli (1700-1782)



Johann II Bernoulli (1710-1790)

Together with, but also independent from their father Johann, the three sons shown above also carried on mathematics and related sciences. So e.g. the results in fluid mechanics of Daniel Bernoulli are still used today in aerodynamics. Johann II's sons became also mathematicians; Johann III worked besides on astronomy also on probability theory.

Resources

[Britannica]: **Bernoulli, Jakob**. (2006). In: Encyclopædia Britannica. Retrieved November 15, 2006, from Encyclopædia Britannica Online, `http://www.britannica.com/eb/article-9078864`

[Dilcher]: A Bibliography of Bernoulli Numbers. (2005). Karl Dilcher, `http://www.mscs.dal.ca/~dilcher/bernoulli.html`

[Gourdon]: **Introduction on Bernoulli's numbers**. (2002). In: Xavier Gourdon, Pascal Sebah: Mathematical constants and computation, `http://numbers.computation.free.fr/Constants/Miscellaneous/bernoulli.html`

[MacTutor]: **Jacob (Jacques) Bernoulli**. (1998). In: John J. O'Connor, Edmund F. Robertson: The MacTutor History of Mathematics archive, `http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Bernoulli_Jacob.html`

[MacTutor2]: **Johann Bernoulli**. (1998). In: John J. O'Connor, Edmund F. Robertson: The MacTutor History of Mathematics archive, `http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Bernoulli_Johann.html`

[Reich]: **Die Familie Bernoulli**. (2000). In: Karin Reich, Große Mathematiker, `http://www.kk.s.bw.schule.de/mathge/bernoull.htm`

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