History of Mathematics **Philipp BRAUN**

Problem Set 9 - The Young Gauß

1. The following theorem characterizes positive integers that can be written as the sum of three squares.
   Theorem (Gauß). *A positive integer can be expressed as a sum of three squares if and only if it is not of the form* $8^k(8m+7)$.
   Prove that this theorem implies the famous result Gauß entered in his mathematical diary on July 10, 1796: $E\Upsilon PEKA!$ $\quad \text{num} = \triangle + \triangle + \triangle.$
   (Every positive integer can be written as the sum of three triangular numbers.)

Proof. According to the theorem, $8m+3$ can be written as the sum of three squares, say x_1, x_2, and x_3. x_1, x_2, and x_3 are all odd, because: if x_i is even, then there is y_i such that $x_i = 2y_i$, which implies $x_i^2 \equiv (2y_i)^2 \equiv 4y_i^2 \equiv 0 \pmod 4$, and if x_i is odd, then there is y_i such that $x_i = 2y_i + 1$, which implies $x_i^2 \equiv (2y_i+1)^2 \equiv 4y_i^2 + 4y_i + 1 \equiv 1 \pmod 4$; since $8m+3 \equiv 3 \pmod 4$, $x_1^2 + x_2^2 + x_3^2 \equiv 3 \pmod 4$, and therefore $x_i^2 \equiv 1 \pmod 4$ for each $i \in \{1,2,3\}$. Hence $x_i = 2y_i + 1$ for some y_i (for each $i \in \{1,2,3\}$), which means $8m+3 = (2y_1+1)^2 + (2y_2+1)^2 + (2y_3+1)^2 = 4(y_1^2+y_2^2+y_3^2) + 4(y_1+y_2+y_3) + 3$. Subtract 3 on both sides and divide by 8 to get

$$m = \frac{y_1^2+y_2^2+y_3^2}{2} + \frac{y_1+y_2+y_3}{2} = \frac{y_1(y_1+1)}{2} + \frac{y_2(y_2+1)}{2} + \frac{y_3(y_3+1)}{2} = t_{y_1} + t_{y_2} + t_{y_3},$$

where t_n denotes the nth triangular number. $\square$

2. By carrying out a portion of the process used by Gauß to show that the regular 17-gon is constructable, find the exact value of $\cos(2\pi/17) + \cos(8\pi/17)$ in terms of radicals. Specifically, begin by noting that 3 is a primitive root modulo 17, and the successive powers of 3 are $1, 3, 9, 10, 13, 5, 15, 11, 16, 14, 8, 7, 4, 12, 2, 6 \pmod{17}$. Let $z = e^{2\pi i/17}$ and consider the following sums: $a_1 = z + z^9 + z^{13} + z^{15} + z^{16} + z^8 + z^4 + z^2$, $a_2 = z^3 + z^{10} + z^5 + z^{11} + z^{14} + z^7 + z^{12} + z^6$, $b_1 = z + z^{13} + z^{16} + z^4$, $b_2 = z^9 + z^{15} + z^8 + z^2$, $b_3 = z^3 + z^5 + z^{14} + z^{12}$, $b_4 = z^{10} + z^{11} + z^7 + z^6$. Justify the following conclusions: $a_1 + a_2 = 1, a_1 a_2 = -4, b_1 b_2 = -1$. Thus determine $b_1 = 2(\cos(2\pi/17) + \cos(8\pi/17))$. For extra credit, carry out the final step and thus find the exact value of $\cos(2\pi/17)$:

$$\frac{-1 + \sqrt{17} + \sqrt{2(17 - \sqrt{17})} + 2\sqrt{17 + 3\sqrt{17} - \sqrt{2(17 - \sqrt{17})} - 2\sqrt{2(17 + \sqrt{17})}}}{16}.$$

Solution.

$$\begin{aligned}
a_1 + a_2 &= z + z^9 + z^{13} + z^{15} + z^{16} + z^8 + z^4 + z^2 + z^3 + z^{10} + z^5 + z^{11} + z^{14} + z^7 + z^{12} + z^6 \\
&= z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8 + z^9 + z^{10} + z^{11} + z^{12} + z^{13} + z^{14} + z^{15} + z^{16} \\
&= \frac{1 - z^{17}}{1 - z} - 1 = -1,
\end{aligned}$$

because $z^{17} = e^{17(2\pi i/17)} = e^{2\pi i} = (-1)^2 = 1$.

$$\begin{aligned}
a_1 a_2 &= (z + z^9 + z^{13} + z^{15} + z^{16} + z^8 + z^4 + z^2)(z^3 + z^{10} + z^5 + z^{11} + z^{14} + z^7 + z^{12} + z^6) \\
&= z^4 + z^{11} + z^6 + z^{12} + z^{15} + z^8 + z^{13} + z^7 + z^{12} + z^{19} + z^{14} + z^{20} + z^{23} + z^{16} + z^{21} \\
&\quad + z^{15} + z^{16} + z^{23} + z^{18} + z^{24} + z^{27} + z^{20} + z^{25} + z^{19} + z^{18} + z^{25} + z^{20} + z^{26} + z^{29} \\
&\quad + z^{22} + z^{27} + z^{21} + z^{19} + z^{26} + z^{21} + z^{27} + z^{30} + z^{23} + z^{28} + z^{22} + z^{11} + z^{18} + z^{13} \\
&\quad + z^{19} + z^{22} + z^{15} + z^{20} + z^{14} + z^7 + z^{14} + z^9 + z^{15} + z^{18} + z^{11} + z^{16} + z^{10} + z^5 \\
&\quad + z^{12} + z^7 + z^{13} + z^{16} + z^9 + z^{14} + z^8 \\
&= [z^{17}(z + z^9 + z^{13}) + z^{15} + z^{16} + z^{17}(z^8 + z^4 + z^2 + z^3 + z^{10} + z^5 + z^{11}) + z^{14} + \\
&\quad z^{17}(z^7 + z^{12} + z^6)] + [z^{17}(z + z^9) + z^{13} + z^{15} + z^{16} + z^8 + z^4 + z^{17}(z^2 + z^3 + z^{10} \\
&\quad + z^5) + z^{11} + z^{14} + z^7 + z^{12} + z^6] + [z^{17}(z) + z^9 + z^{13} + z^{15} + z^{16} + z^8 + z^{17}(z^4 \\
&\quad + z^2 + z^3 + z^{10} + z^5) + z^{11} + z^{14} + z^7 + z^{12} + z^{17}(z^6)] + [z^{17}(z) + z^9 + z^{13} + z^{15} \\
&\quad + z^{16} + z^{17}(z^8 + z^4 + z^2 + z^3) + z^{10} + z^5 + z^{11} + z^{14} + z^7 + z^{12} + z^{17}(z^6)] \\
&= 4(a_1 + a_2) = -4,
\end{aligned}$$

and similarly,

$$\begin{aligned}
b_1b_2 &= (z+z^{13}+z^{16}+z^4)(z^9+z^{15}+z^8+z^2)\\
&= z^{10}+z^{16}+z^9+z^3+z^{22}+z^{28}+z^{21}+z^{15}+z^{25}+z^{31}+z^{24}+z^{18}+z^{13}+z^{19}\\
&\quad +z^{12}+z^6\\
&= z^{17}(z+z^2)+z^3+z^{17}(z^4+z^5)+z^6+z^{17}(z^7+z^8)+z^9+z^{10}+z^{17}(z^{11})+z^{12}\\
&\quad +z^{13}+z^{17}(z^{14})+z^{15}+z^{16}\\
&= -1.
\end{aligned}$$

Using the same method, we get

$$\begin{aligned}
b_3b_4 &= (z^3+z^5+z^{14}+z^{12})(z^{10}+z^{11}+z^7+z^6)\\
&= z^{13}+z^{14}+z^{10}+z^9+z^{15}+z^{16}+z^{12}+z^{11}+z^{24}+z^{25}+z^{21}+z^{20}+z^{22}+z^{23}\\
&\quad +z^{19}+z^{18}\\
&= z^{17}(z+z^2+z^3+z^4+z^5+z^6+z^7+z^8)+z^9+z^{10}+z^{11}+z^{12}\\
&\quad +z^{13}+z^{14}+z^{15}+z^{16}\\
&= -1,
\end{aligned}$$

which we will need later. Trivially, $b_1+b_2=a_1$, and $b_3+b_4=a_2$.

The fact that $a_1+a_2=-1$, and $a_1a_2=-4$ implies that a_1 and a_2 solve $x^2+x-4=0$, which means that

$$\{a_1,a_2\}=\{-\frac{1}{2}-\frac{\sqrt{17}}{2},-\frac{1}{2}+\frac{\sqrt{17}}{2}\}.$$

Since $a_1+a_2=-1$, and $re(z+z^{15}+z^8+z^9)>re(z^6+z^7+z^{10}+z^{11})$ (by geometry; draw a 17-gon with center at (0,0) and one vertex at (1,0)), $re(z^2)>re(z^3)$, $re(z^4)>re(z^5)$, $re(z^{15})>re(z^{14})$, and $re(z^{13})>re(z^{12})$, $a_1>a_2$, which means $a_1=-\frac{1}{2}+\frac{\sqrt{17}}{2}$ and $a_2=-\frac{1}{2}-\frac{\sqrt{17}}{2}$. Thus b_1 and b_2 solve $x^2-a_1x-1=0$, hence

$$\begin{aligned}
\{b_1,b_2\} &= \left\{\frac{a_1}{2}-\sqrt{\frac{a_1^2}{4}+1},\frac{a_1}{2}+\sqrt{\frac{a_1^2}{4}+1}\right\}\\
&= \left\{-\frac{1}{4}+\frac{1}{4}\sqrt{17}-\sqrt{\frac{1}{16}-\frac{1}{8}\sqrt{17}+\frac{1}{16}17+1},\right.\\
&\qquad \left.-\frac{1}{4}+\frac{1}{4}\sqrt{17}+\sqrt{\frac{1}{16}-\frac{1}{8}\sqrt{17}+\frac{1}{16}17+1}\right\}\\
&= \left\{-\frac{1}{4}+\frac{1}{4}\sqrt{17}-\sqrt{\frac{17}{8}-\frac{1}{8}\sqrt{17}},-\frac{1}{4}+\frac{1}{4}\sqrt{17}+\sqrt{\frac{17}{8}-\frac{1}{8}\sqrt{17}}\right\}.
\end{aligned}$$

Since $re(z) > re(z^2), re(z^4) > re(z^8), re(z^{13}) > re(z^9)$, and $re(z^{16}) > re(z^{15})$ (by geometry again), $b_1 > b_2$, i.e.

$$b_1 = -\frac{1}{4} + \frac{1}{4}\sqrt{17} + \sqrt{\frac{17}{8} - \frac{1}{8}\sqrt{17}}\}.$$

On the other hand, it is true that

$$b_1 = z + z^{13} + z^{16} + z^4 = e^{2\pi i/17} + e^{13(2\pi i/17)} + e^{16(2\pi i/17)} + e^{4(2\pi i/17)}.$$

Using the fact that $e^{ix} = \cos(x) + i\sin(x)$, and observing that

$$\cos(A) + \cos(B) = 2\cos(\frac{A+B}{2})\cos(\frac{A-B}{2})$$

and

$$\sin(A) + \sin(B) = 2\sin(\frac{A+B}{2})\sin(\frac{A-B}{2})$$

are identities results in

$$\begin{aligned}
b_1 &= \cos\left(\frac{2\pi}{17}\right) + \cos\left(\frac{2\cdot 13\cdot \pi}{17}\right) + \cos\left(\frac{2\cdot 16\cdot \pi}{17}\right) + \cos\left(\frac{2\cdot 4\cdot \pi}{17}\right) + \\
&\quad i(2\sin\left(\frac{2(1+16)\pi}{17}\right)\cos\left(\frac{2(1-16)\pi}{17}\right) + 2\sin\left(\frac{2(3+14)\pi}{17}\right)\cos\left(\frac{2(3-14)\pi}{17}\right)) \\
&= \cos\left(\frac{2\pi}{17}\right) + \cos\left(\frac{2\cdot 13\cdot \pi}{17}\right) + \cos\left(\frac{2\cdot 16\cdot \pi}{17}\right) + \cos\left(\frac{2\cdot 4\cdot \pi}{17}\right) + \\
&\quad i(2\sin(2\pi)\cos\left(\frac{2(1-16)\pi}{17}\right) + 2\sin(2\pi)\cos\left(\frac{2(3-14)\pi}{17}\right)) \\
&= \cos\left(\frac{2\pi}{17}\right) + \cos\left(\frac{2\cdot 4\cdot \pi}{17}\right) + \cos\left(\frac{2\cdot 1\cdot \pi}{17}\right) + \cos\left(\frac{2\cdot 4\cdot \pi}{17}\right) \\
&= 2\cos\left(\frac{(1+1)\pi}{17}\right)\cos\left(\frac{(1-1)\cdot \pi}{17}\right) + 2\cos\left(\frac{(4+4)\cdot \pi}{17}\right)\cos\left(\frac{(4-4)\pi}{17}\right) \\
&= 2\cos\left(\frac{(1+1)\pi}{17}\right)\cos\left(\frac{(1-1)\cdot \pi}{17}\right) + 2\cos\left(\frac{(4+4)\cdot \pi}{17}\right)\cos\left(\frac{(4-4)\pi}{17}\right) \\
&= 2(\cos\left(\frac{2\pi}{17}\right) + \cos\left(\frac{8\pi}{17}\right)),
\end{aligned}$$

where the third "=" is true because $\cos(\alpha) = \cos(2\pi - \alpha)$.

Putting these facts together gives us the desired value:

$$\cos\left(\frac{2\pi}{17}\right) + \cos\left(\frac{8\pi}{17}\right) = -\frac{1}{8} + \frac{1}{8}\sqrt{17} + \sqrt{\frac{17}{32} - \frac{1}{32}\sqrt{17}}.$$

Let $c_1 = z + z^{16}, c_2 = z^{13} + z^4$. Since $b_3 + b_4 = a_2$, we get, using the same arguments as above, that

$$b_3 = -\frac{1}{4} - \frac{1}{4}\sqrt{17} + \sqrt{\frac{17}{8} + \frac{1}{8}\sqrt{17}}\}.$$

Since $c_1c_2 = (z + z^{16})(z^{13} + z^4) = z^{14} + z^5 + z^{29} + z^{20} = z^{17}(z^3) + z^5 + z^{14} + z^{17}(z^{12}) = b_3$, and $c_1 + c_2 = b_1$, we get, by observing that c_1 and c_2 solve $x^2 - b_1x + b_3 = 0$,

$$\begin{aligned}
\{c_1, c_2\} &= \left\{\frac{b_1}{2} - \sqrt{\frac{b_1^2}{4} - b_3}, \frac{b_1}{2} + \sqrt{\frac{b_1^2}{4} - b_3}\right\} \\
&= \left\{-\frac{1}{8} + \frac{1}{8}\sqrt{17} + \sqrt{\frac{17}{32} - \frac{1}{32}\sqrt{17}} - \sqrt{\frac{17}{16} + \frac{3}{16}\sqrt{17} - \sqrt{\frac{85}{128} + \frac{19}{128}\sqrt{17}}},\right. \\
&\quad \left. -\frac{1}{8} + \frac{1}{8}\sqrt{17} + \sqrt{\frac{17}{32} - \frac{1}{32}\sqrt{17}} + \sqrt{\frac{17}{16} + \frac{3}{16}\sqrt{17} - \sqrt{\frac{85}{128} + \frac{19}{128}\sqrt{17}}}\right\}.
\end{aligned}$$

By comparing $re(z)$ and $re(z^{16})$ with $re(z^4)$ and $re(z^{13})$, we get $c_1 > c_2$, i.e.

$$c_1 = -\frac{1}{8} + \frac{1}{8}\sqrt{17} + \sqrt{\frac{17}{32} - \frac{1}{32}\sqrt{17}} + \sqrt{\frac{17}{16} + \frac{3}{16}\sqrt{17} - \sqrt{\frac{85}{128} + \frac{19}{128}\sqrt{17}}}.$$

But c_1 is nothing else than $e^{2\pi i/17} + e^{2\cdot 16\cdot\pi i/17}$, which simplifies to $2\cos(2\pi/17)$, using again that $\cos(\alpha) = \cos(2\pi - \alpha)$, and that $\sin(\alpha) = -\sin(2\pi - \alpha)$:

$$\begin{aligned}
e^{2\pi i/17} + e^{2\cdot 16\cdot\pi i/17} &= \cos\left(\frac{2\pi}{17}\right) + \cos\left(\frac{2\cdot 16\cdot\pi}{17}\right) + i(\sin\left(\frac{2\pi}{17}\right) + \sin\left(\frac{2\cdot 16\cdot\pi}{17}\right)) \\
&= 2\cos\left(\frac{2\pi}{17}\right).
\end{aligned}$$

Therefore $\cos(2\pi/17) = c_1/2$, which can also be written as

$$\frac{-1 + \sqrt{17} + \sqrt{2(17 - \sqrt{17})} + 2\sqrt{17 + 3\sqrt{17} - \sqrt{2(17 - \sqrt{17})} - 2\sqrt{2(17 + \sqrt{17})}}}{16},$$

because $\sqrt{2(85 + 19\sqrt{17})} = \sqrt{2(17 - \sqrt{17})} + 2\sqrt{2(17 + \sqrt{17})}$ since

$$\begin{aligned}
&\left(\sqrt{2(17 - \sqrt{17})} + 2\sqrt{2(17 + \sqrt{17})}\right)^2 \\
&= 2(17 - \sqrt{17}) + 8\sqrt{(17 - \sqrt{17})(17 + \sqrt{17})} + 8(17 + \sqrt{17}) \\
&= 10 \cdot 17 + 6 \cdot \sqrt{17} + 8\sqrt{17^2 - 17} = 10 \cdot 17 + 6 \cdot \sqrt{17} + 32\sqrt{17} = 170 + 38\sqrt{17} \\
&= 2(85 + 39\sqrt{17}) = \left(2(85 + 39\sqrt{17})\right)^2.
\end{aligned}$$

3. (a) Use Gauß's test to determine for which p the series $\sum_{k=1}^{\infty}\left(\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots(2k)}\right)^{p}$ converges.

Solution.

(b) Using Gauß's ${}_2F_1$ summation $F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$, $c\neq 0,-1,-2,\ldots$, $c>a+b$, evaluate the combinatorial sum $\sum_{k=0}^{n}\binom{2n}{2k}\binom{2n-2k}{n-k}\binom{2k}{k}$.

Solution.